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The effect of rotation on the simpler modes of motion of a liquid in an elliptic paraboloid

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The six simplest modes of motion are considered and three rotational effects investigated:

(i) The effect of the rotation of the earth.
(ii) The effect of the rotation of the container.
(iii) The effect of the rotation of the liquid within the container.

The first two are shown to be equivalent for motion in a paraboloid, and the last two are also equivalent when the paraboloid is *circular*. In the case of an elliptic paraboloid the last is rather more difficult and one must first derive a solution of the non-linear equations representing 'elliptic rotation' and then consider deviations from it.

The changes in frequency consequent on the rotation are derived in all three cases for all six modes. In the case of the earth's rotation the disposition and character of the amphidromic (nodal) points and the amphidromic waves that rotate round these points are investigated in detail. One mode is particularly interesting because it has four amphidromic points, the waves rotate in a positive sense around two of these and in a negative sense round the other two.

1. Introduction

Previous treatments of the effect of the earth's rotation on shallow water motions in other than axially symmetric basins have generally led to a great deal of involved theory, see for example Taylor (1920), Proudman (1928), Corkan & Doodson (1952) and recently Van Dantzig & Lauwerier (1962). The elliptic paraboloid seems to be the simplest of such basins to deal with because the solutions are *polynomials* in the spatial co-ordinates and the frequency equation is algebraic instead of transcendental (this was pointed out by Goldsbrough 1930 in the case of no rotation). Furthermore, the elliptic paraboloid is at least as good an approximation to many naturally occurring bodies of water as is, for instance, a rectangle or ellipse of constant depth favoured by earlier theorists. The disposition and character of the 'tidal' waves and amphidromic (nodal) points in such a basin, and the effect of rotation on the frequencies of oscillation have immediate practical relevance in the study of actual lakes (see Platzman & Rao 1963) and possibly also in the study of partially enclosed seas.

The lower modes are likely to be of the most importance from a practical point of view, since they are the ones most easily excited, and they are also the easiest to treat theoretically. Accordingly we consider in detail only the two lower groups of modes; these are the 'displacement' modes that involve a motion of the centre of gravity of the liquid and in which the horizontal velocities are functions of time only (they have been described in part previously, see Ball 1963*a*, *b*) and the 'deformation' modes that involve uniform rotation, expansion and distortion of the liquid and in which the spatial derivatives of the velocities are functions of time only. We could, of course, extend the analysis and consider the next group of modes in which the velocities are quadratic functions of the spatial co-ordinates, the *second* derivatives then being functions of time only.

Three rotational effects are considered:

(i) The effect of the earth's rotation; this introduces Coriolis terms without corresponding centripetal terms.

(ii) The effect of rotation of the elliptic container; this introduces both Coriolis terms and centripetal terms.

(iii) The effect of the rotation of the liquid relative to the elliptic container.

The first two of these are essentially equivalent (for motion in a paraboloid); results for one can be converted into results for the other merely by redefining certain constants (a relationship which could be useful for the application of laboratory experiments). The last is rather more difficult and one must first derive a solution of the non-linear equations representing 'elliptic rotation' and then consider deviations from it. It is a problem that does not arise when one is considering axially symmetric basins ((ii) and (iii) are then equivalent) and does not appear to have been discussed previously, but is of some geophysical interest since most naturally occurring bodies of water do in fact possess horizontal circulations (usually wind driven).

2. Formulation

The equation of the underlying surface is

$$z = \tfrac{1}{2}(\alpha x^2 + \beta y^2), \tag{2.1}$$

the equation of continuity, for a liquid of depth h, is

$$Dh/Dt + h(\partial u/\partial x + \partial v/\partial y) = 0, \tag{2.2}$$

and the equations of motion in the absence of rotation are

$$Du/Dt + g(\partial h/\partial x + \alpha x) = 0 \tag{2.3}$$

and

$$Dv/Dt + g(\partial h/\partial y + \beta y) = 0. \tag{2.4}$$

If we take account of the earth's rotation then Coriolis terms must be introduced without corresponding centripetal terms, since the latter are absorbed in the value for $\mathfrak{g}$. The equations of motion then become

$$Du/Dt + g(\partial h/\partial x + \alpha x) = fv, \tag{2.5}$$

and

$$Dv/Dt + g(\partial h/\partial y + \beta y) = -fu, \tag{2.6}$$

where f is the Coriolis parameter of the earth's rotation and is equal to $2\Omega \sin\theta$; θ being the latitude and Ω the angular velocity of the earth. However, if the basin is rotating with constant angular velocity ω and axes are taken which rotate with the basin, then centripetal and Coriolis terms are present and the equations of motion become

$$Du/Dt+g[\partial h/\partial x+(\alpha-\omega^2/g)x] = 2\omega v, \tag{2.7}$$

and

$$Dv/Dt+g[\partial h/\partial y+(\beta-\omega^2/g)y] = -2\omega u. \tag{2.8}$$

Equations (2.7) and (2.8) become identical to equations (2.5) and (2.6) if we make the transformation

$$\alpha-\omega^2/g\to\alpha, \quad \beta-\omega^2/g\to\beta, \quad 2\omega\to f. \tag{2.9}$$

Any result derived from equations (2.5) and (2.6) can therefore always be reinterpreted as a result for (2.7) and (2.8) (and vice versa). In particular we cannot have a static solution for (2.5) and (2.6), with a finite volume of liquid, if either α or β is not positive; correspondingly we cannot have a static solution for (2.7) and (2.8) if either $\alpha g \leqslant \omega^2$ or $\beta g \leqslant \omega^2$. In the former case the liquid collapses under gravity and in the latter under the influence of centrifugal forces.

3. Displacement modes

It was shown by Ball (1963*a*) that the *exact* solution of the non-linear equations (2.2), (2.5) and (2.6) for simple displacement modes is given by

$$u = dX/dt, \quad v = dY/dt, \tag{3.1}$$

and

$$h = H-\tfrac{1}{2}[\alpha(x-X)^2+\beta(y-Y)^2]; \tag{3.2}$$

where H is a constant and X and Y (the co-ordinates of the centre of gravity of the liquid) are functions of time only which satisfy the ordinary linear differential equations

$$d^2X/dt^2+\alpha gX = fdY/dt, \tag{3.3}$$

and

$$d^2Y/dt^2+\beta gY = -fdX/dt. \tag{3.4}$$

The frequencies of these modes (ν) are therefore given by

$$(\nu^2-\alpha g)(\nu^2-\beta g) = \nu^2 f^2, \tag{3.5}$$

a result that is independent of the volume of liquid in the basin. This equation has the same form as the approximate frequency equation derived by Lamb (1932, § 212*a*) for the lowest modes of a rotating rectangular basin of uniform depth.

In the absence of rotation the frequencies of these modes are $(\alpha g)^{\frac{1}{2}}$ and $(\beta g)^{\frac{1}{2}}$, each mode involving displacement parallel to one of the principal axes of the elliptical container with the other principal axis as a nodal line† (as indicated schematically in figure 1). Each mode also exerts an oscillatory transverse force on the container and can be generated by transverse oscillation with appropriate frequency or by a sudden transverse motion of the container.

The earth's rotation does not alter the product of the roots of (3.5), one must therefore be increased and the other decreased in frequency. In fact we find that the frequency of the low frequency mode is *decreased* and the liquid particles instead of moving parallel to the major axis now move in ellipses whose major axes are parallel to that of the basin. The direction of motion is anticlockwise (in

† These are true nodal lines only when the amplitude is small; the non-linear terms give a small oscillation in depth of twice the frequency of the normal mode.

the northern hemisphere) and the nodal line is replaced by a 'positive' amphidromic point at the centre of the ellipse (i.e. about which the cotidal lines and amphidromic wave rotate in a positive sense). This type of motion is observed for the low frequency mode in Lake Erie (see Platzman & Rao 1963). Opposite conditions prevail for the high frequency mode; its frequency is *increased*, the particles instead of moving parallel to the minor axis of the basin now move *clockwise* in ellipses whose major axes are parallel to the minor axis of the basin.

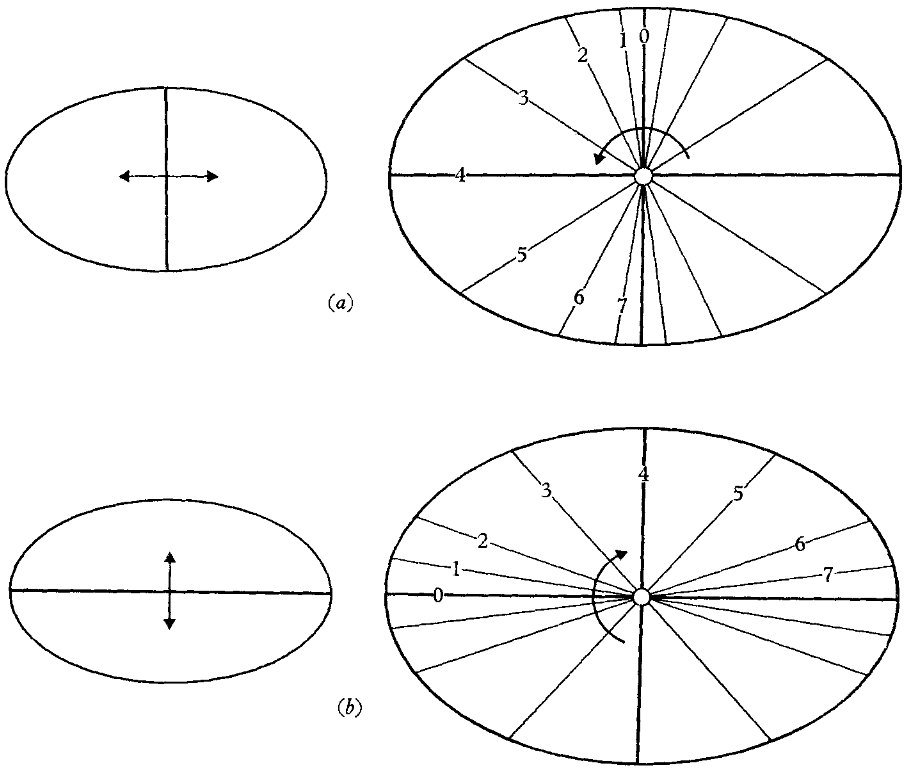

FIGURE 1. Displacement modes: (*a*) low frequency (longitudinal) mode; (*b*) high frequency (transverse) mode. In each case the small ellipse gives the nodal line in the absence of rotation and the large ellipse gives the sequence of 'instantaneous nodal lines' when the system is rotating anticlockwise. The direction of rotation of these lines round the central amphidromic point is indicated by an arrow.

The nodal line is replaced by a 'negative' amphidromic point (see figure 1). Both of these modes now have an oscillatory angular momentum and exert an oscillatory couple on the basin with a frequency of twice that of the normal mode.

It is of interest to compare these simple results with previous work. The change in frequencies is in agreement with the rule of 'repulsion of frequencies' (Rayleigh 1903). The existence of a negative amphidromic point is in disagreement with Taylor's (1920) conjecture that amphidromic points should always be positive. It is, however, in agreement with later work by Jeffreys (1925), and the whole pattern of co-tidal lines is qualitatively very similar to those derived by Goldstein

(1929) for analogous modes of motion in an elliptic basin of constant depth, though the present analysis is a great deal less complicated.

In the case of a circular basin, without rotation, the normal modes are to some extent arbitrary and there are no unique normal modes to be obtained by allowing both the ellipticity of the container and the rotation to tend to zero. If we first let $f \to 0$ and secondly let $\alpha \to \beta$, then the modes we obtain are simply two transverse oscillations at right angles to one another; whereas if we let $\alpha \to \beta$ and then let $f \to 0$, the modes we obtain are two rotations of opposite sense. This means that the modes of a nearly circular basin are very sensitive to changes in the degree of rotation and ellipticity.

If the basin is rotating with angular velocity ω, then by applying the transformation (2.9), we find the appropriate frequency equation

$$(\nu^2 - \alpha g + \omega^2)(\nu^2 - \beta g + \omega^2) = 4\nu^2\omega^2, \tag{3.6}$$

and the situation is substantially the same as in the previous case. On the other hand the *rotation of the liquid within the container has no effect on the frequency equation* because it was shown by Ball (1963*a*) that equations (3.3) and (3.4) are independent of *any* (shallow water) motion that the liquid may possess relative to its centre of gravity.

4. The deformation modes

It is clear from the form of the non-linear equations (2.2), (2.5) and (2.6) that it is possible to find exact solutions in which the velocities are linear functions, and the depth is a quadratic function of x and y; the coefficients of the polynomials being functions of time. The derivatives of the velocities with respect to x and y will then be functions of time only, i.e. the deformation, and in particular the vorticity and expansion, will be uniform. The linearized perturbation equations are a great deal simpler than the full equations and are adequate to determine the effect of the earth's rotation or the rotation of the basin on the deformation modes. It is only in the last case to be considered, that is the rotation of the liquid within the basin, that we need consider the non-linear equations and this is dealt with in § 7.

The perturbation equations corresponding to equations (2.2), (2.5) and (2.6) can be written

$$\partial\eta/\partial t + \partial(h_0 u)/\partial x + \partial(h_0 v)/\partial y = 0, \tag{4.1}$$

$$\partial u/\partial t + g\,\partial\eta/\partial x = fv, \tag{4.2}$$

$$\partial v/\partial t + g\,\partial\eta/\partial y = -fu, \tag{4.3}$$

where we have put $h = h_0 + \eta$ and h_0 is the static equilibrium depth given by

$$h_0 = H_0 - \tfrac{1}{2}(\alpha x^2 + \beta y^2). \tag{4.4}$$

We now search for solutions of the form

$$u = A_1 x + B_1 y, \tag{4.5}$$

$$v = A_2 x + B_2 y, \tag{4.6}$$

and

$$\eta = \eta_0 - \tfrac{1}{2}(ax^2 + 2cxy + by^2), \tag{4.7}$$

where $A_1, B_1, A_2, B_2, \eta_0, a, b$ and c are functions of time only. If these expressions are substituted into equations (4.1)–(4.3) and coefficients equated, we obtain 8 simultaneous, first-order, ordinary linear differential equations in the 8 unknowns:

$$\left.\begin{aligned} &\partial\eta_0/\partial t+(A_1+B_2)H_0 = 0, \quad \partial a/\partial t+(3A_1+B_2)\alpha = 0,\\ &\partial b/\partial t+(A_1+3B_2)\beta = 0, \quad \partial c/\partial t+B_1\alpha+A_2\beta = 0,\\ &\partial A_1/\partial t-ag = fA_2, \quad \partial B_1/\partial t-cg = fB_2,\\ &\partial A_2/\partial t-cg = -fA_1, \quad \partial B_2/\partial t-bg = -fB_1. \end{aligned}\right\} \tag{4.8}$$

It is rather more informative when investigating the geometry of the motions described by equations (4.8), to consider the variables

$$\left.\begin{aligned} \chi &= \partial u/\partial x+\partial v/\partial y = A_1+B_2 \quad \text{(expansion)},\\ \zeta &= \partial v/\partial x-\partial u/\partial y = A_2-B_1 \quad \text{(vorticity)},\\ \left.\begin{aligned} L &= \partial u/\partial x-\partial v/\partial y = A_1-B_2\\ M &= \partial v/\partial x+\partial u/\partial y = A_2+B_1 \end{aligned}\right\} &\quad \text{(distortion)}. \end{aligned}\right\} \tag{4.9}$$

The interpretation of χ as the rate of increase of horizontal area of a fluid element and of ζ as twice the angular velocity of an element is well known. The components L, M of the distortion are less familiar but also have a simple geometric interpretation. L is a measure of the rate at which a fluid element is becoming elliptical, the principal axes of the ellipse being parallel to the axes of co-ordinates, M is a similar measure, the axes of the ellipse now being at 45 degrees to the axes of co-ordinates.

By eliminating η_0, a, b and c from (4.8) and using the new independent variables (4.9) we find

$$d\zeta/dt+f\chi = 0, \tag{4.10}$$

$$d^2\chi/dt^2+2\chi(\alpha+\beta)g+L(\alpha-\beta)g-f\,d\zeta/dt = 0, \tag{4.11}$$

$$d^2L/dt^2+2\chi(\alpha-\beta)g+L(\alpha+\beta)g-f\,dM/dt = 0, \tag{4.12}$$

and

$$d^2M/dt^2+M(\alpha+\beta)g-\zeta(\alpha-\beta)g+f\,dL/dt = 0, \tag{4.13}$$

and the four equations (4.10)–(4.13) now contain only four dependent variables ζ, χ, L and M. If we seek a solution of frequency ν we find

$$\begin{aligned} \nu\{[\nu^4-\nu^2 3(\alpha+\beta)g+8\alpha\beta g^2]\,[\nu^2-(\alpha+\beta)g]\\ -2f^2[\nu^4-\nu^2 2(\alpha+\beta)g+2\alpha\beta g^2]+\nu^2 f^4\} = 0. \end{aligned} \tag{4.14}$$

This equation gives the frequencies of the (four) modes of uniform deformation and is (like the corresponding equation for the displacement modes) independent of the volume of liquid in the basin.

Let us first consider what happens when there is no rotation (i.e. $f = 0$), we then have the four roots

$$\nu_1^2 = \tfrac{1}{2}g\{3(\alpha+\beta)-[(\alpha+\beta)^2+8(\alpha-\beta)^2]^{\frac{1}{2}}\}, \tag{4.15}$$

$$\nu_2^2 = \tfrac{1}{2}g\{3(\alpha+\beta)+[(\alpha+\beta)^2+8(\alpha-\beta)^2]^{\frac{1}{2}}\}, \tag{4.16}$$

$$\nu_3^2 = g(\alpha+\beta), \tag{4.17}$$

and

$$\nu_4 = 0. \tag{4.18}$$

The first three of these modes were described by Goldsbrough (1930). The last one ($\nu = 0$), which is necessarily rotational, does not seem to have been described previously except in the case of a *circular* paraboloid (Miles & Ball 1963). We will subsequently (§ 7) be particularly concerned with the non-linear solution corresponding to this mode.

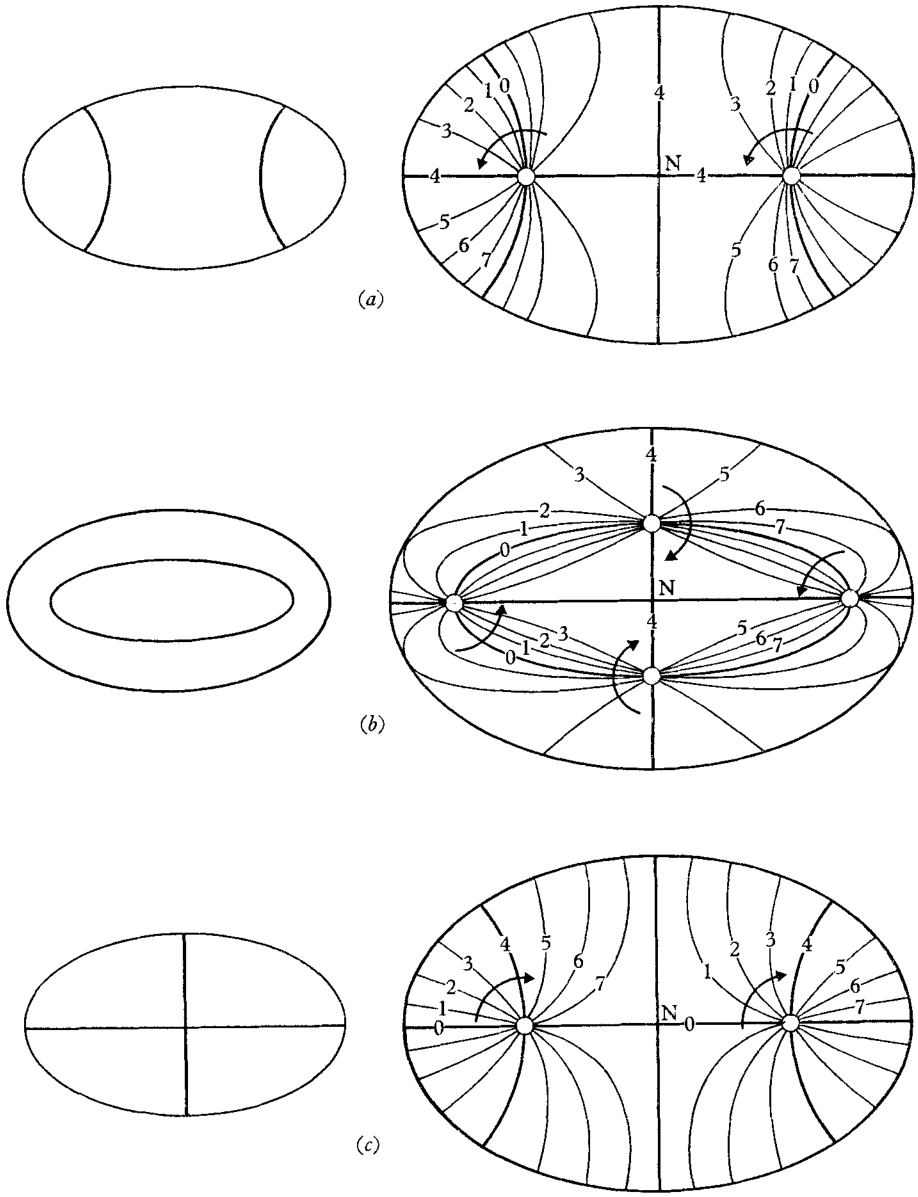

FIGURE 2. Oscillatory deformation modes: (*a*) low-frequency 'distortional' mode; (*b*) high-frequency 'expansional' mode; (*c*) high-frequency 'distortional' mode. In each case the small ellipse gives the nodal lines in the absence of rotation and the large ellipse gives the sequence of 'instantaneous nodal lines' when the system is rotating anticlockwise. The direction of rotation of these lines round the amphidromic points is indicated by arrows.

In the first two of these modes we have, from equations (4.10) and (4.13), with f equal to zero,

$$\zeta = 0 \quad \text{and} \quad M = 0,$$

whence $$c = 0 \quad \text{(from equations (4.8))}.$$

The elliptical periphery of the liquid therefore always has its principal axis parallel to those of the basin. One of these modes (the low-frequency one (4.15)) is predominantly 'distortional'; the minor axis of the liquid contracts as the major axis expands. The nodal lines are hyperbolic arcs intersecting the major axis of the ellipse (see figure 2). On the other hand the high-frequency mode (4.16) is predominantly 'expansional'; the principal axes of the liquid expand and contract together and the nodal line is a complete ellipse of greater eccentricity than the periphery of the liquid. In the case of extreme ellipticity the distinction between these modes on the basis of their degree of distortion and expansion is no longer very significant. The low-frequency mode becomes almost entirely longitudinal, with two nodes, and the high-frequency mode becomes largely transverse with the nodal ellipse almost touching the periphery at the ends of the major axis. There is consequently a region of almost stagnant liquid at each 'end' of the ellipse. Neither of these modes possess angular momentum and there is no couple exerted by or on the container.

The third mode (4.17) is purely distortional, since equations (4.9)–(4.13) imply that ζ, χ, L, η_0, a and b are all zero, leaving M and c as the only non-vanishing dependent variables. The principal axes of the ellipse are now the nodal lines and, to a first approximation, the elliptical periphery of the liquid does not change shape but executes an oscillatory rotation about its centre. This mode is interesting because, unlike the others, it is associated with an oscillatory angular momentum and a corresponding oscillatory couple exerted on (or by) the basin (provided $\alpha \neq \beta$). In the case of a circular basin the first and third modes become similar (as remarked by Goldsbrough 1930).

The fourth mode is unique (among those here discussed) because it involves a perturbation of the vorticity ζ; however, χ, L, η_0, a, b and c are all zero. This mode, in fact, is merely an infinitesimal 'elliptic rotation' without any change in shape of the liquid. It cannot be generated except by some process that can generate vorticity (usually involving friction). In §7 we shall determine the change in shape of the liquid when the elliptic rotation is finite and subsequently determine the effect of such a rotation on the frequencies of the other modes.

5. The effect of the earth's rotation

The effect of the earth's rotation on these modes, or on corresponding modes in other types of basin, does not seem to have been described previously (except for some discussion of the change in frequency); a fairly detailed description is therefore given. Because of the simplicity of the solutions it is possible to do this without numerical computation though the algebra is a little tedious. To determine the effect of rotation on the frequencies it is convenient to express the frequency equation (4.14) in dimensionless form. We define the dimensionless frequency σ by

$$\sigma^2 = \nu^2/(\alpha g + \beta g), \tag{5.1}$$

and two dimensionless constants

$$N^2 = f^2/(\alpha g + \beta g), \tag{5.2}$$

and

$$Q^2 = 1 + 8(\alpha+\beta)^2/(\alpha+\beta) \quad (1 \leqslant Q < 3). \tag{5.3}$$

The constant N is a measure of the degree of rotation and, for realistic values of α and β, is usually less than 10^{-1}. The constant Q is a measure of the ellipticity of the basin and ranges from unity, when the basin is circular, to a limiting value of three when the basin is extremely elliptical.

If the root $\nu = 0$, whose value is unchanged by rotation, is omitted, equation (4.14) in dimensionless form becomes

$$F = [\sigma^4 - 3\sigma^2 + \tfrac{1}{4}(9-Q^2)][\sigma^2 - 1] - N^2[2\sigma^4 - 4\sigma^2 + \tfrac{1}{8}(9-Q^2)] + \sigma^2 N^4 = 0, \tag{5.4}$$

and the three roots, when $N = 0$, are

$$\sigma_1^2 = \tfrac{1}{2}(3-Q), \quad \sigma_2^2 = \tfrac{1}{2}(3+Q), \quad \sigma_3^2 = 1. \tag{5.5}$$

Using the Descartes rule of signs and observing that

$$\operatorname{sgn} F = \quad -, \quad +, \quad +, \quad -, \quad +$$

for

$$\sigma^2 = 0, \quad \tfrac{1}{2}(3-Q), \quad 1, \quad \tfrac{1}{2}(3+Q), \quad \infty,$$

we infer that the roots of (5.4) lie in the intervals $(0, \tfrac{1}{2}(3-Q))$, $(1, \tfrac{1}{2}(3+Q))$ and $(\tfrac{1}{2}(3+Q), \infty)$. The two higher frequency roots are therefore *increased* in frequency by the earth's rotation whereas the lowest frequency root is *decreased* in frequency again in agreement with Rayleigh's rule of repulsion of frequencies.

When the ellipticity is large by comparison with the rotation, i.e. when

$$Q - 1 \gg N, \tag{5.6}$$

it is easy to show that the approximate roots of (5.4) are

$$\sigma_1^2 = \tfrac{1}{2}(3-Q) - \tfrac{1}{4}N^2(3Q+1)(3-Q)/(Q^2-Q), \tag{5.7}$$

$$\sigma_2^2 = \tfrac{1}{2}(3+Q) + \tfrac{1}{4}N^2(3Q-1)(3+Q)/(Q^2+Q), \tag{5.8}$$

and

$$\sigma_3^2 = 1 + \tfrac{1}{2}N^2(7+Q^2)/(Q^2-1), \tag{5.9}$$

in agreement with the qualitative conclusions reached in the preceding paragraph. This approximation for the high frequency root σ_2 is valid for the whole range of Q and in fact gives exact values for the root when $Q = 1$ or when $Q \to 3$. The approximations for the other two roots are clearly invalid when the inequality (5.6) is not satisfied. A rather unwieldy approximation for these roots, valid for small N over the whole range of Q, can be found by using the known approximation for the high frequency root σ_2 and determining the sum and product of the other two roots from the coefficients of (5.4). We then find

$$\sigma_1^2 + \sigma_3^2 = \tfrac{1}{2}(5-Q) + \tfrac{1}{4}N^2(5Q^2+3)/(Q^2+Q), \tag{5.10}$$

$$\sigma_1^2\sigma_3^2 = \tfrac{1}{4}[3-Q][2 + N^2(Q-1)^2/(Q^2+Q)], \tag{5.11}$$

and the roots are therefore given by

$$\sigma^2 = \tfrac{1}{4}\{5 - Q + \tfrac{1}{2}N^2(5Q^2+3)/(Q^2+Q) \\ \pm [(Q-1)^2 + N^2(3+Q)(1+8Q-Q^2)/(Q^2+Q)]^{\frac{1}{2}}\}. \tag{5.12}$$

The properties of these modes, mentioned in the previous section, are modified by the presence of rotation. For instance, the first two modes, which in the absence of rotation have zero values of c, M and ζ, no longer have this property; there is a small amplitude (dependent on f) oscillation of these quantities. Furthermore, the angular momentum of the liquid also oscillates in value, implying an oscillatory couple acting on the basin. In the third mode the quantities ζ, χ, L, η_0, a and b are no longer zero but again oscillate with an amplitude dependent on f.

These three modes, instead of having nodal lines, have two or four amphidromic points, the positions of which may be determined by setting $\partial\eta/\partial t$ equal to zero. Differentiating (4.7) with respect to time and eliminating η_0, a, b and c by using equations (4.8) we obtain

$$d\eta/dt = -\chi H_0 + \tfrac{1}{2}\{\alpha x^2(2\chi + L) + xy[M(\alpha+\beta) - \zeta(\alpha-\beta)] + \beta y^2(2\chi - L)\}. \quad (5.13)$$

The quantities M and ζ, which occur in the coefficient of xy in (5.13), differ in phase by $\frac{1}{2}\pi$ from χ and L; consequently $d\eta/dt$ cannot vanish at all times unless both xy and the sum of the remaining terms are zero. We can express χ in terms of L by using (4.10) and (4.11) to give

$$\chi[f^2 + 2(\alpha+\beta)g - \nu^2] = L(\beta-\alpha)g, \quad (5.14)$$

and we suppose that the major axis of the ellipse is in the x direction so that the coefficient of L on the right-hand side of (5.14) is positive. The condition for the vanishing of the χ, L terms in (5.13) is therefore

$$\tfrac{1}{2}\alpha x^2(f^2 - \nu^2 + 4\beta g) + \tfrac{1}{2}\beta y^2(\nu^2 - f^2 - 4\alpha g) = (\beta - \alpha) H_0 g, \quad (5.15)$$

and the amphidromic points occur at the points of intersection of the conic (5.15) with the axes ($xy = 0$).

In the case of the first mode we have shown that the frequency is *reduced* by the earth's rotation and is therefore less than ν_1 of equation (4.15); so that in the coefficient of x^2 in (5.15) we have

$$f^2 - \nu^2 + 4\beta g > 4\beta g - \nu_1^2 = \tfrac{1}{2}g\{5\beta - 3\alpha + [(\alpha+\beta)^2 + 8(\beta-\alpha)^2]^{\frac{1}{2}}\}, \quad (5.16)$$

which is clearly positive ($\beta > \alpha$). Similarly for the coefficient of y^2

$$\nu^2 - f^2 - 4\alpha g < \nu_1^2 - 4\alpha g = \tfrac{1}{2}g\{3\beta - 5\alpha - [(\alpha+\beta)^2 + 8(\beta-\alpha)^2]^{\frac{1}{2}}\}, \quad (5.17)$$

which is clearly negative. The conic (5.15) is therefore a hyperbola that intersects the x-axis; furthermore, the points of intersection, as may easily be shown, always lie within the liquid. The first mode therefore has *two* amphidromic points on the major axis of the container, which incidentally are *positive* as in the case of the low-frequency displacement mode (see figure 2 (a)). There is now a double 'tide' rotating positively round the periphery of the basin. In the limiting case of a circular basin the two amphidromic points coalesce at the centre to produce a double positive amphidromic point.

In the case of the second mode, where the frequency is increased by rotation, we can show by similar reasoning that the conic (5.15) is an ellipse which intersects the axes at four points that always lie within the liquid. There are therefore *four* amphidromic points of which the two on the y-axis are *negative* (in agreement with the orbital motion of the liquid particles), whereas the two on the x-axis are anomalously *positive* (see figure 2 (*b*)), thus providing a simple example of a single mode with amphidromic points of both senses. In the limiting case of a circular basin the amphidromic points are replaced by a circular nodal line.

In the case of the third mode, where the frequency is again increased by rotation, the conic (5.15) may be a hyperbola that intersects the x-axis or an ellipse which, however, intersects the y-axis outside the region occupied by the liquid. There are therefore, as in the case of the first mode, just two amphidromic points. These points are now *negative* so that there is a double tide that rotates negatively round the periphery of the liquid (see figure 2 (*c*)). In the limiting case of a circular basin the two amphidromic points coalesce at the centre of the basin to produce a double negative amphidromic point.

Each of these modes has a null point (marked N in figure 2) at the centre of the basin, where both the velocity and the slope of the free surface vanish at all times. Topological considerations suggest that adjacent amphidromic points of the same sign must always be separated by a null point and adjacent amphidromic points not so separated must be of opposite sign.

These three modes have another important property in common, namely that each liquid column has the same potential vorticity $(\zeta+f)/h$ during the course of its motion as it would have if at rest in its equilibrium position. This is an immediate consequence of the conservative property of potential vorticity

$$D[(\zeta+f)/h]\,Dt = 0,$$

the 'perturbation' expression for which, in this particular case, is

$$d[\zeta-f\eta_0/H_0]/dt = 0,$$

and since we are considering a purely oscillatory solution we must have

$$\zeta-f\eta_0/H_0 = 0.$$

These modes can therefore be generated by simple processes that do not involve transfer of vorticity to the liquid, for instance, appropriate variation of (atmospheric) pressure on the free surface or an appropriate motion of the elliptic basin (earthquake).

The fourth mode, of zero frequency, is again an infinitesimal 'elliptic rotation' relative to the basin; it also involves a slight spreading of the liquid if the rotation is 'cyclonic' (i.e. ζ and f have the same sign) and a slight contraction if 'anticyclonic', as is evident from equations (4.8). The perturbation of the potential vorticity at the centre of the liquid is $\zeta-f\eta_0/H_0$ and we also have

$$4\eta_0/H = a/\alpha+b/\beta$$

by virtue of volume conservation. Then from (4.8) we find

$$\zeta-f\eta_0/H_0 = \zeta\{1+f^2/[2g(\alpha+\beta)]\},$$

which is never zero. This mode can therefore be generated only by processes which transfer vorticity to the liquid.

6. The effect of rotation of the basin

The corresponding results for a rotating basin are perhaps most easily derived by transforming equation (4.14) in the way indicated in (2.9). We then obtain

$$[\nu^4 - 3\nu^2 g(\alpha+\beta) + 8g^2\alpha\beta][\nu^2 - g(\alpha+\beta)] \\ + 4\omega^2[\nu^2 - 2g(\alpha+\beta)][\omega^2 - g(\alpha+\beta)] = 0, \tag{6.1}$$

or in dimensionless form, using the quantities defined by equations (5.1)–(5.3) (with $f = 2\omega$), we obtain

$$[\sigma^4 - 3\sigma^2 + \tfrac{1}{4}(9-Q^2)][\sigma^2-1] - N^2(1-\tfrac{1}{4}N^2)(\sigma^2-2) = 0. \tag{6.2}$$

We need consider only the case where

$$\tfrac{1}{8}(9-Q^2) > N^2(1-\tfrac{1}{4}N^2) > 0, \tag{6.3}$$

for if this condition is not satisfied static equilibrium is impossible (i.e. either $\alpha g < \omega^2$ or $\beta g < \omega^2$, as may be seen by returning to the definitions of Q^2 (5.3) and N^2(5.2)). Using the Descartes rule of signs we can then make exactly the same inferences about (6.2) as we did about (5.4).

When the inequality (5.6) is also satisfied the approximate roots of (6.2) can be written

$$\sigma_1^2 = \tfrac{1}{2}(3-Q) - N^2(Q+1)/(Q^2-Q), \tag{6.4}$$

$$\sigma_2^2 = \tfrac{1}{2}(3+Q) + N^2(Q-1)/(Q^2+Q), \tag{6.5}$$

$$\sigma_3^2 = 1 + 4N^2/(Q^2-1), \tag{6.6}$$

where, as before, the approximate form for σ_2^2 is valid for the whole range of Q whereas the other results are limited to the conditions under which the inequality (5.6) is satisfied. In particular, when Q is unity (i.e. the basin is circular) the frequency of high-frequency mode (σ_2) is independent of rotation (as previously pointed out by Miles & Ball 1963).

Once again we can determine an approximate formula for σ_1 and σ_3 which is valid over the whole range of Q by using the approximation for σ_2 (equation (6.5)) and deducing from the coefficients of equation (6.2) that

$$\sigma_1^2 + \sigma_3^2 = \tfrac{1}{2}(5-Q) - N^2(Q-1)/(Q^2+Q), \tag{6.7}$$

and

$$\sigma_1^2\sigma_3^2 = \tfrac{1}{2}(3-Q) - N^2(3Q-1)/(Q^2+Q), \tag{6.8}$$

whence

$$\sigma^2 = \tfrac{1}{4}\{5 - Q - 2N^2(Q-1)/(Q^2+Q) \pm [(Q-1)^2 + 4N^2(Q^2+6Q+1)/(Q^2+Q)]^{\frac{1}{2}}\}. \tag{6.9}$$

The various properties of these modes are very similar to those of the modes described in the preceding section and will not be discussed further.

7. Non-linear solution with uniform deformation

The main object of this section is the determination of an exact solution of the non-linear equations (2.2), (2.5) and (2.6) representing elliptic rotation. This solution is given by equations (7.11), (7.12) and (7.14)–(7.17) and is all that is required in § 8; however, for completeness, it is indicated briefly how more general non-linear solutions can be found. If we substitute the expressions (4.5)–(4.7) for u, v and η ($\eta = h - h_0$, see equation (4.4)) into the non-linear equations (2.2), (2.5) and (2.6), and equate coefficients, we again obtain 8 simultaneous, first-order, ordinary differential equations for the 8 unknowns, the equations now being non-linear. They are

$$\left.\begin{aligned} dA_1/dt - ag - fA_2 + A_1^2 + A_2B_1 &= 0, \\ dB_1/dt - cg - fB_2 + A_1B_1 + B_1B_2 &= 0, \\ dA_2/dt - cg + fA_1 + A_1A_2 + A_2B_2 &= 0, \\ dB_2/dt - bg + fB_1 + A_2B_1 + B_2^2 &= 0, \\ d\eta_0/dt + H_0(A_1+B_2) + \eta_0(A_1+B_2) &= 0, \\ da/dt + \alpha(3A_1+B_2) + a(3A_1+B_2) + 2cA_2 &= 0, \\ db/dt + \beta(A_1+3B_2) + b(A_1+3B_2) + 2cB_1 &= 0 \\ \text{and}\qquad dc/dt + \alpha B_1 + \beta A_2 + aB_1 + bA_2 + 2c(A_1+B_2) &= 0. \end{aligned}\right\} \tag{7.1}$$

The first three terms in each equation are linear in the dependent variables and are the same as the terms of equations (4.8), the additional terms are all non-linear (quadratic). It is a simple matter to determine numerically the solution of this set of equations with any given initial conditions. Furthermore, these equations have three integrals expressing constancy of volume, energy and peripheral circulation (angular momentum is not constant because of the possibility of the elliptical container exerting a couple in the liquid, as has been remarked previously).

It is more convenient and informative to use the variables defined by (4.9) together with

$$s = \alpha + \beta + a + b, \quad r = \alpha - \beta + a - b, \quad q = 2c; \tag{7.2}$$

in terms of these variables equations (7.1) become

$$d\zeta/dt + \chi\zeta + f\chi = 0, \tag{7.3}$$

$$d\chi/dt + \tfrac{1}{2}(\chi^2 + L^2 + M^2 - \zeta^2) - sg + (\alpha+\beta)g - f\zeta = 0, \tag{7.4}$$

$$dL/dt + \chi L - rg + (\alpha - \beta)g - fM = 0, \tag{7.5}$$

$$dM/dt + \chi M - qg + fL = 0, \tag{7.6}$$

$$d\eta_0/dt + \chi(H_0 + \eta_0) = 0, \tag{7.7}$$

$$ds/dt + 2\chi s + Lr + Mq = 0, \tag{7.8}$$

$$dr/dt + 2\chi r + Ls + \zeta q = 0, \tag{7.9}$$

$$dq/dt + 2\chi q + Ms - \zeta r = 0. \tag{7.10}$$

Various special solutions of these equations can easily be found and these will be discussed elsewhere; we are at present interested only in the steady-state solution. On this assumption equation (7.7) implies that

$$\chi = 0, \tag{7.11}$$

since the central depth, $H_0+\eta_0$, cannot be zero. Equation (7.3) is then automatically satisfied and (7.5) and (7.6) become

$$rg = (\alpha-\beta)g - fM, \tag{7.12}$$

and

$$qg = fL. \tag{7.13}$$

Equation (7.8) now reduces to $L(\alpha-\beta)g = 0,$

whence

$$L = 0 \quad (\text{since } \alpha \neq \beta), \tag{7.14}$$

$$q = 0, \tag{7.15}$$

and equation (7.9) is automatically satisfied. Equations (7.4) and (7.10) reduce to

$$\tfrac{1}{2}(M^2-\zeta^2) - sg + (\alpha+\beta)g - f\zeta = 0, \tag{7.16}$$

and

$$Ms = \zeta r, \tag{7.17}$$

which, together with (7.12), if we suppose the vorticity to be given, suffice to determine s, r and M. Eliminating r and M we obtain a cubic in $sg+f\zeta$ which, for realistic values of ζ, has only one real root; there is therefore a unique steady solution (elliptic rotation) with given vorticity. If the container is circular ($\alpha = \beta$), then $M = 0$, and we obtain the more familiar case of circular rotation, with the spread of the liquid consequent on its rotation being given by

$$sg = 2\alpha g - f\zeta - \tfrac{1}{2}\zeta^2 \quad (\text{from (7.16)});$$

decrease in s corresponds to an increase in spread of the liquid. If the container is not rotating ($f = 0$) then $r = \alpha - \beta$ and the free surface of the liquid is a *circular* paraboloid, despite the fact that it rotates within an *elliptic* basin. The curvature of this paraboloid is, however, somewhat less than that produced by circular rotation with the same vorticity.

8. The effect of finite elliptic rotation

We now investigate the effect of rotation of the liquid within the elliptic basin on the frequencies of the other deformation modes, assuming that the basin itself is not rotating and neglecting the rotation of the earth.

If we take elliptic rotation as our basic solution, indicated by suffix 0, and consider perturbations from it, equations (7.3) to (7.10) become

$$d\zeta'/dt + \chi\zeta_0 = 0, \tag{8.1}$$

$$d\chi/dt + M'M_0 - \zeta'\zeta_0 - s'g = 0, \tag{8.2}$$

$$dL/dt - r'g = 0, \tag{8.3}$$

$$dM'/dt + \chi M_0 - qg = 0, \tag{8.4}$$

$$d\eta_0'/dt + \chi H_0 = 0, \tag{8.5}$$

$$ds'/dt + 2\chi s_0 + Lr_0 + qM_0 = 0, \tag{8.6}$$

$$dr'/dt + 2\chi r_0 + Ls_0 + q\zeta_0 = 0, \tag{8.7}$$

and

$$dq/dt + s'M_0 + M's_0 - r'\zeta_0 - \zeta' r_0 = 0. \tag{8.8}$$

These equations have a steady state (zero frequency) solution for which χ, L, r' and q are zero. This merely represents an infinitesimal change in the degree of elliptic rotation and corresponds to the fourth mode of our previous solutions. If we disregard this solution the remaining modes of uniform deformation are determined by

$$d^2\chi/dt^2 + 2\chi(\alpha+\beta)g + L(\alpha-\beta)g + 2qM_0 g = 0, \tag{8.9}$$

$$d^2L/dt^2 + 2\chi(\alpha-\beta)g + Ls_0 g + q\zeta_0 g = 0, \tag{8.10}$$

and

$$d^2q/dt^2 + L[s_0\zeta_0 - M_0(\alpha-\beta)] + q[2(\alpha+\beta)g - s_0 g] = 0. \tag{8.11}$$

These equations may be derived by eliminating ζ', M', s' and r' from (8.1)–(8.3) and (8.5)–(8.8) and simplifying slightly by using equations (7.12), (7.16) and (7.17) with $f = 0$. The frequency equation is therefore given by

$$\begin{vmatrix} 2(\alpha+\beta)g-\nu^2 & (\alpha-\beta)g & 2M_0 g \\ 2(\alpha-\beta)g & s_0 g-\nu^2 & \zeta_0 g \\ 0 & s_0\zeta_0 - M_0(\alpha-\beta) & 2(\alpha+\beta)g - s_0 g - \nu^2 \end{vmatrix} = 0. \tag{8.12}$$

On the assumption that $\zeta_0^2 \ll (\alpha+\beta)g$, the approximate solution of (7.12), (7.16) and (7.17) is

$$M_0^2 = \zeta_0^2\left(\frac{\alpha+\beta}{\alpha-\beta}\right) = \zeta_0^2\tfrac{1}{8}(Q^2-1), \tag{8.13}$$

and

$$s_0 = [\alpha+\beta][1 - N^2\tfrac{1}{16}(9-Q^2)], \tag{8.14}$$

where N^2 is now defined by $\quad N^2 = \zeta_0^2/(\alpha g+\beta g)$, (8.15)

and Q is as defined previously (5.3). Using these values in expanding the determinant (8.12) and expressing the result in dimensionless form, we find

$$F = [\sigma^4 - 3\sigma^2 + \tfrac{1}{4}(9-Q^2)][\sigma^2-1] - \tfrac{1}{8}N^2(9-Q^2)[\sigma^2 - 2 + \tfrac{3}{8}(Q^2-1)] = 0. \tag{8.16}$$

Observing that when

$\sigma^2 =$	$0,$	$\tfrac{1}{2}(3-Q),$	$\tfrac{1}{6}[8-(1+Q^2)^{\frac{1}{2}}],$	$1,$	$\tfrac{1}{2}(3+Q),$	$\infty,$	
sng F is	$-,$	$+,$	$+,$	$+,$	$-,$	$+,$	$(\tfrac{11}{3} > Q^2 > 1),$
or	$-,$	$+,$	$+,$	$-,$	$-,$	$+,$	$(\tfrac{49}{9} > Q^2 > \tfrac{11}{3}),$
or	$-,$	$-,$	$+,$	$-,$	$-,$	$+,$	$(9 > Q^2 > \tfrac{49}{9}),$

we infer that in the range of ellipticities $\tfrac{11}{3} > Q^2 > 1$ the mode of lowest frequency is decreased and the other two modes are increased in frequency by elliptic rotation (this rotational effect is qualitatively the same as in the previous cases). In the range $\tfrac{49}{9} > Q^2 > \tfrac{11}{3}$ the mode of highest frequency is increased and the other two modes are decreased in frequency, whereas in the range of large ellipticity, $9 > Q^2 > \tfrac{49}{9}$, the middle mode is decreased and the other two modes are increased in frequency. The approximate roots, provided the inequality (5.6) is satisfied, are

$$\sigma_1^2 = \tfrac{1}{2}(3-Q) - N^2(9-Q^2)(Q+1)(7-3Q)/[32Q(Q-1)], \tag{8.17}$$

$$\sigma_2^2 = \tfrac{1}{2}(3+Q) + N^2(9-Q^2)(Q-1)(7+3Q)/[32Q(Q+1)], \tag{8.18}$$

$$\sigma_3^2 = 1 + N^2(9-Q^2)(11-3Q^2)/[16(Q^2-1)]. \tag{8.19}$$

An approximate result analogous to (5.12) or (6.9) can also be derived.

9. Higher modes

We can investigate the nth group of higher modes by assuming that h is an nth degree polynomial in x and y, and that the velocities are $(n-1)$th degree polynomials. By substituting into the *perturbation* form of equations (2.2), (2.5) and (2.6) and equating coefficients we obtain $3n+1$ equations involving the $3n+1$ coefficients of the highest degree terms in the polynomials. These equations form a determinate set on their own and lead to a $(3n+1)$th degree frequency equation. The vorticity is an $(n-2)$th degree polynomial in x and y, and, in the absence of rotation, conservation of vorticity implies that the vorticity is zero for oscillatory modes and constant (with respect to time) for the zero frequency modes. There are $n-1$ terms of degree $n-2$ in the polynomial for the vorticity, indicating that there are $n-1$ independent modes of zero frequency, most simply defined by supposing that each mode involves the vanishing of all except one of these coefficients. The remaining $2n+2$ roots of the frequency equation occur in equal pairs of opposite sign corresponding to $n+1$ oscillatory modes. *In the absence of rotation we therefore have* $n+1$ *oscillatory modes with no vorticity and* $n-1$ *modes with constant vorticity and zero frequency.*

When rotation is introduced the behaviour of the system depends to some extent on the parity of n. If n is odd, the $n-1$ modes, formerly of zero frequency, occur as equal pairs of opposite sign to produce $\frac{1}{2}(n-1)$ oscillatory modes, giving a total of $\frac{1}{2}(3n+1)$ oscillatory modes. If n is even, then $n-2$ of the modes, formerly of zero frequency, occur as equal pairs of opposite sign to produce $\frac{1}{2}(n-2)$ oscillatory modes, giving a total of $\frac{3}{2}n$ oscillatory modes. The remaining mode still has zero frequency, this mode corresponds to elliptic rotation when $n = 2$.

Some of the preceding remarks are exemplified by the cubic modes ($n = 3$). When there is no rotation there are four oscillatory modes and two modes of zero frequency, the latter modes leaving the free surface undisturbed. The nodal lines of each oscillatory mode consist of one or other of the principal axes of the basin together with either an ellipse or a hyperbola, the four possible combinations corresponding to the four modes. The modes of zero frequency each consist of two equal and opposite circulation cells, in one case separated by the minor axis and in the other case by the major axis of the basin.

When the system is rotating the frequency equation is

$$[\nu^2(6\alpha g+\beta g+f^2-\nu^2)(\alpha g+6\beta g+f^2-\nu^2)-\alpha\beta g^2f^2][(3\alpha g-\nu^2)(3\beta g-\nu^2)-\nu^2f^2] + 36(\beta-\alpha)^2\alpha\beta g^4\nu^2 = 0$$

and the two modes formerly of zero frequency, appear as a pair whose frequencies are equal but of opposite sign, corresponding to a single oscillatory mode of arbitrary phase. This additional mode is of low frequency ($\nu < f$) and therefore of the second class (see Lamb 1932, § 223); it still leaves the free surface almost undisturbed. Very good approximation to this and higher second-class modes can be obtained when the rotation is small by assuming that there is a rigid upper boundary to the liquid. Attention is thereby confined to the second-class modes, because the first-class modes depend for their existence on gravitational restoring forces resulting from the deformation of the free surface, whereas the second-class modes merely require a gradient of potential vorticity within the liquid. The second-class modes will be discussed in more detail in a later paper where it will be shown that elliptic rotation is *unstable* in some circumstances.

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